Answers and Hints for Homework

1. Extend EM to G such that MG = FM, and connect BG. Then △BMG ≅ △CMF, so BG = CF and ∠G = ∠MFC. Since ∠BAD = ∠DAC and ME // AD, it follows that ∠BAD = ∠AEM and ∠DAC = ∠MFC, hence ∠BEG = ∠G. Thus BE = BG = CF. Because BE = AB + AE and AE = AF, we have 2BE = AB + AE + CF = AB + AF + CF = AB + AC. Therefore, BE = (AB + AC)/2.

2. Let P and Q be the midpoints of AC and AD respectively. Connect PB, PM, QE, and QM. Since P, M, Q are the midpoints of the three sides of △ACD, we have AP = MQ and AQ = PM. Given ∠ABC = ∠AED = 90° , it follows that AP = BP and AQ = EQ, so BP = MQ and PM = QE. Moreover, ∠BPC = 2∠BAP = 2∠QAE = ∠DQE, and ∠CPM = ∠CAD = ∠MQD. Thus ∠BPM = ∠MQE, and △BPM ≅ △MQE. Therefore, MB = ME.

3. For definiteness, assume △ABC is rotated clockwise about point C by an angle θ (0° < θ < 90°); other cases can be discussed similarly. Let K and M be the midpoints of B'C and BC respectively. Since B'K = BC = CM, B'A' = BA = CD, and ∠B' = ∠B = ∠DCM, we have △CMD ≅ △B'KA'. Thus A'K = DM and ∠CMD = ∠B'KA'. In the isosceles triangle, ∠MKC = ∠KMC, so ∠A'KM = ∠DMQ (where Q lies on the extension of KM). Since A'K = DM, the distances from A' and D to line MK are equal, and hence the midpoint of A'D lies on KM. This proves the conclusion for this position of △A'B'C. If point K lies on line MD, then ∠A'KM = 0° or 180° , and the proposition still holds.

4. Extend BD to P such that DP = DB, and extend CE to Q such that EQ = EC. Connect AP, AQ, PC, and QB. In △ABQ and △APC, by SAS, we can prove AB = AP, AC = AQ, and ∠PAC = 2∠DAB + ∠BAC = 2∠CAE + ∠BAC = ∠BAQ. Thus △ABQ ≅ △APC, so PC = BQ. Since MD and ME are the midlines of △BPC and △BQC respectively, MD = PC and ME = BQ. Therefore, MD = ME.

5. Let N and P be the midpoints of BO and CO respectively. Then KN = AO = BO = LP, LN = CO = DO = MP, and ∠KNL = ∠AOC = 120° + ∠BOC = ∠BOD = ∠LPM. Thus △KNL ≅ △LPM, so KL = LM and ∠PLM = ∠NKL. When ∠BOC ⩽ 60° , ∠KLM = ∠NLP + ∠KLN + ∠PLM = ∠BNL + ∠KLN + ∠NKL = 180° − ∠BNK = 60° . When ∠BOC > 60° , ∠KLM = ∠NLP − ∠KLN − ∠PLM = ∠BNL − ∠

KLN − ∠NKL = 180° − ∠BNK = 60° . Hence △KLM is an isosceles triangle with vertex angle 60° , i.e., an equilateral triangle.

6. Connect AB and AE. Since AB // DE, ∠BAE = ∠AED. Noting AB = ED and AE is common, △ABE ≅ △EDA. Thus ∠AEB = ∠EAD, so AD // BE, and ∠DAC = ∠ACE. Further, since ∠AMD = ∠CME and MA = MC, △AMD ≅ △CME. Therefore DM = ME, i.e., M is the midpoint of DE.

7. Construct ∠DAE = ∠BCA, and draw DE // BC intersecting AE at E. By ASA, △ABC ≅ △EAD. Thus AE = BA = AC, ∠DAE = ∠ACB = 40° , so ∠EAC = 100° − 40° = 60° . Hence △AEC is equilateral, so EC = AE = DE and ∠BCE = 20° . Therefore ∠CDE = ∠DCE = ∠BCD = ∠BCE = 10° .

8. Draw the angle bisector CE of ∠C intersecting AM at E. In △CAE and △BCN, since AC = BC, ∠NCB = ∠EAC, and ∠B = ∠ACE = 45° , △CAE ≅ △BCN. Thus NB = EC. In △MNB and △MEC, since BM = CM, ∠B = ∠ECM = 45° , and NB = EC, △MNB ≅ △MEC. Therefore ∠NMB = ∠CMA.

9. On the exterior of side BC, construct ∠BCD = ∠MCA and ∠CBD = ∠CAM, with CD and BD intersecting at D. Since AC = BC, △CBD ≅ △CAM. Thus BD = AM = m and ∠DCM = 90° , CD = CM. Also ∠NCD = ∠MCD − ∠MCN = 90° − 45° = 45° = ∠MCN. Since CN is common, △MCN ≅ △DCN, so ND = MN = 1. Given ∠NBC = 45° and ∠CBD = ∠A = 45° , then ∠NBD = 90° . Hence △NBD is a right triangle with ∠NBD = 90° , BD = m, BN = n, ND = 1.

10. Let P and Q be the midpoints of AC and AD. Connect PB, PF, QE, and QF. Then PB = PA = FQ, QE = QA = FP. Since FB = FE, △FPB ≅ △EQF. Thus ∠FPC + ∠CPB = ∠FQD + ∠DQE. Also ∠CPF = ∠CAD = ∠FQD, so ∠CPB = ∠DQE. Since ∠CPB = 2∠BAC and ∠DQE = 2∠QAE, we have ∠BAC = ∠QAE. Given ∠BAE = 70° and ∠DAC = 30° , then ∠BAC = 20° .

11. Connect CD. In △BDF and △BDC, BF = AB = BC, ∠DBF = ∠DBC, and BD is common, so △BDF ≅ △BDC. Thus ∠BFD = ∠BCD. In △BDC and △ADC, DB = DA, BC = AC, and CD = CD, so △BDC ≅ △ADC. Hence ∠BCD = ∠ACD = ∠C, and therefore ∠BFD = 30° .

12. Extend AE and BC to intersect at F. In right △ACF and right △BCD, since ∠CAF = ∠CBD and AC = BC, Rt△ACF ≅ Rt△BCD. Thus AF = BD. Since AE = BD = AF = EF, in Rt△ABE and Rt△FBE, ∠ABE = ∠FBE. Therefore BD is the angle bisector of ∠ABC.

13. Extend BC to F such that CF = DE, and connect AF. Since BE = AC, Rt△BDE ≅ Rt△AFC. Thus AF = BD, and ∠DBE = ∠CAF, so ∠BAF = ∠BAC + ∠CAF = ∠BAC + ∠ABC = 90° . In Rt△ABF, AF = BD, BF = BC + CF = BC + DE = 1. Hence ∠ABC = 30° .

14. Since OA = OC and $\angle C_1OA = \angle A_1OC$, $Rt\triangle AOC_1 \cong Rt\triangle COA_1$, so $AC_1 = A_1C$. Noting AC is common, $Rt\triangle AA_1C \cong Rt\triangle CC_1A$, hence $AA_1 = CC_1$. Thus $Rt\triangle BCC_1 \cong Rt\triangle BAA_1$, and therefore AB = BC.